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Second-order expansion of indices in the generalised Villain model

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Abstract. Critical indices of several operators in the generalised Villain model are calculated to second order along a critical line emerging from the model's multicritical point. Relations between these expansions support the identification suggested between this line and a critical line of the Ashkin–Teller model.

1. Expansion of critical indices

Kadanoff (1977, 1979) has suggested an identification of the critical behaviours in the generalised Villain and Ashkin–Teller models. A correspondence between several of the operators in each model was proposed and a first-order expansion was made of the relevant generalised Villain model critical indices. The expansion parameter represented the introduction of equal amounts of spin waves (square symmetry breaking excitations) and vortices into the system. These first-order expansions satisfied (to first order) all the algebraic relations believed to hold between the corresponding critical indices in the Ashkin–Teller model. In this paper, we continue the expansions to second order in the hope of retaining these algebraic relations and thereby establishing the identification of both operators and critical behaviour on a firmer footing.

We ask what happens to the critical indices of two degenerate ($x_1 = x_2$) local operators, $\mathrm{O}_1(r)$ and $\mathrm{O}_2(r)$, when the marginal ($x = d = 2$) operator $\mathrm{E}(r)$ is added to the Hamiltonian.

$$\exp H \to \exp(H + \lambda \int \mathrm{d}r\, \mathrm{E}(r)).$$

This makes the average of any operator $\tilde{\mathrm{O}}(r_1, \ldots, r_n)$ change by

$$\langle \tilde{\mathrm{O}} \rangle \to \left\langle \tilde{\mathrm{O}}\left(1 + \lambda \int \mathrm{d}r\, \mathrm{E}(r) + \tfrac{1}{2}\lambda^2 \int \mathrm{d}r\, \mathrm{d}r' (\mathrm{E}(r)\mathrm{E}(r') - \langle \mathrm{E}(r)\mathrm{E}(r')\rangle) + \cdots \right)\right\rangle.$$

Say O_1 and O_2 are chosen so that at $\lambda = 0$

$$\langle \mathrm{O}_\alpha(r_1)\mathrm{O}_\beta(r_2)\rangle = \delta_{\alpha\beta}/|r_1 - r_2|^{2x_\alpha(\lambda=0)} \qquad (1)$$

$$\alpha, \beta = 1, 2; \qquad |r_1 - r_2| \text{ large.}$$

To retain this relation for $\lambda > 0$ one must allow for mixing of the degenerate operators and/or changes in their normalisation (Wegner 1972, 1976).

$$\mathrm{O}_\alpha(r) \to \mathrm{O}_\alpha(r) + \lambda \sum_{i=1}^{2} A_\alpha^i \mathrm{O}_i(r) + \ldots.$$

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So upon choosing $\tilde{O}(r_1, r_2) = O_\alpha(r_1)O_\beta(r_2)$ the relation (1) becomes

$$\bigg\langle\Big(O_\alpha(r_1)+\lambda \sum_i A^i_\alpha O_i(r_1)\Big)\Big(O_\beta(r_2)+\lambda \sum_j A^j_\beta O_j(r_2)\Big)\Big[1+\lambda \int dr_3\, E(r_3) + \tfrac{1}{2}\lambda^2 \int dr_3\, dr_4 (E(r_3)E(r_4) - \langle E(r_3)E(r_4)\rangle)\Big]\bigg\rangle$$
$$= \frac{\delta_{\alpha\beta}}{|r_1 - r_2|^{2x_\alpha(0)}}\Big[1 - 2\lambda \frac{dx_\alpha}{d\lambda} \ln|r_1 - r_2| + \lambda^2\Big(2\Big(\frac{dx_\alpha}{d\lambda}\Big)^2 \ln^2|r_1 - r_2| - \frac{d^2x_\alpha}{d\lambda^2} \ln|r_1 - r_2|\Big)\Big]. \quad (2)$$

In equation (2), $|r_1 - r_2|^{-2x_\alpha(\lambda)}$ has been Taylor expanded in λ and only those terms relevant to a second-order calculation have been included. Again, note that (2) holds only asymptotically for $|r_1 - r_2|$ large.

Now specialise to the operators of interest in the generalised Villain model. Details of the model and its notation can be found in Kadanoff (1979) and Pruisken and Kadanoff (1980). To specify the cutoff, choose $L(r)$ to be $\ln(r/a)$ when $r \geqslant a$ and 0 otherwise. The marginal operator of interest is $E = a^{-2}E_2$. First consider the operators

$$O_1(r) = (1/\sqrt{2a})(O_{\frac{1}{2},0}(r) + O_{-\frac{1}{2},0}(r))$$
$$O_2(r) = (i/\sqrt{2a})(O_{\frac{1}{2},0}(r) - O_{-\frac{1}{2},0}(r)).$$

These two operators have been identified with P and CR, the polarisation and cross-over operators of the Ashkin–Teller model. When $\lambda = 0$, $x_1 = x_2 = \frac{1}{2}$. Using the integrals

$$\int \langle O_\alpha(r_1)O_\beta(r_2)E(r_3)\rangle\, dr_3 = \frac{\delta_{\alpha\beta}(-1)^\alpha}{|r_1 - r_2|}(-2\pi \ln|r_1 - r_2| + (2\pi \ln a - \pi) + \ldots)$$
$$\int \langle O_\alpha(r_1)O_\beta(r_2)E(r_3)E(r_4)\rangle_c\, dr_3\, dr_4 = \frac{\delta_{\alpha\beta}}{|r_1 - r_2|}(4\pi^2 \ln^2|r_1 - r_2| - (8\pi^2 \ln a) \ln|r_1 - r_2| + \ldots)$$

one may conclude, using (2), that

$$A^1_1 = -A^2_2 = \pi \ln a - \pi/2 \qquad A^1_2 = A^2_1 = 0$$
$$dx_1/d\lambda = -dx_2/d\lambda = -\pi \qquad d^2x_1/d\lambda^2 = d^2x_2/d\lambda^2 = 2\pi^2.$$

So, to second order, the physically interesting critical exponents are:

$$x_1(\lambda) = \tfrac{1}{2} - \pi\lambda + \pi^2\lambda^2$$
$$x_2(\lambda) = \tfrac{1}{2} + \pi\lambda + \pi^2\lambda^2. \quad (3)$$

Now we analyse the critical indices of two more operators

$$O_1(r) = \frac{1}{\sqrt{2}}(F_{11}(r) + a^{-2}E_3(r))$$
$$O_2(r) = \frac{1}{\sqrt{2}}(F_{11}(r) - a^{-2}E_3(r)).$$

These operators have been identified with ϵ' and ϵ, an irrelevant operator and the operator conjugate to $T-T_c$ in the Ashkin–Teller model. When $\lambda=0$, $x_1=x_2=2$. Using the integrals

$$\int \langle O_\alpha(r_1)O_\beta(r_2)E(r_3)\rangle\,dr_3$$

$$=(\delta_{\alpha\beta}(-1)^\alpha/|r_1-r_2|^4)[8\pi\ln|r_1-r_2|+(\pi-8\pi\ln a)+\ldots]$$

$$\int \langle O_\alpha(r_1)O_\beta(r_2)E(r_3)E(r_4)\rangle_c\,dr_3\,dr_4$$

$$=(\delta_{\alpha\beta}/|r_1-r_2|^4)[64\pi^2\ln^2|r_1-r_2|-(16\pi^2+128\pi^2\ln a)\ln|r_1-r_2|]$$
$$+((-1)^{\alpha+\beta}/|r_1-r_2|^4)16\pi^2\ln|r_1-r_2|+\ldots$$

one may conclude, again using (2), that

$$A_1^1=-A_2^2=\tfrac{1}{2}\pi-4\pi\ln a \qquad A_2^1=-A_1^2=-\tfrac{1}{2}\pi$$
$$dx_1/d\lambda=-dx_2/d\lambda=4\pi \qquad d^2x_1/d\lambda^2=d^2x_2/d\lambda^2=8\pi^2.$$

So, to second order, two more of the physically interesting critical exponents are:

$$x_1(\lambda)=2+4\pi\lambda+4\pi^2\lambda^2$$
$$x_2(\lambda)=2-4\pi\lambda+4\pi^2\lambda^2. \tag{4}$$

We report here that a similar expansion has been made of the critical index of the operator E_2. This operator has been identified with E, the marginal operator of the Ashkin–Teller model. The result is that $x_{E_2}=2$ to second order.

We can now check to see if our expansions (3) and (4) satisfy the relations expected of Ashkin–Teller critical indices. These are:

(1) $x_E=2$ Definition of marginality.

(2) $x_p=x_{\epsilon/4}$ This relation was derived for the eight-vertex model independently by Baxter and Kelland (1974) and Luther and Peshel (1975). Enting (1975) suggested this relation could be extended to the Ashkin–Teller model. It was independently derived for the Ashkin–Teller model by Kadanoff and Brown (1979).

(3) $x_{CR}=1/x_\epsilon$, $x_{\epsilon'}=4/x_\epsilon$ These relations were derived for the Ashkin–Teller model by Kadanoff and Brown (1979).

Using the mentioned operator identifications, all these relations may be verified to hold (up to second order) for the generalised Villain model critical line under consideration. This leads to the conclusion that Kadanoff's identification of both operators and critical behaviour is most likely correct.

2. Evaluation of the integrals

We would like to say a few words about the evaluation of the integrals in § 1. All of the correlation functions in the generalised Villain model can be written in terms of the two functions (Pruisken and Kadanoff 1980)

$$L(r)=-\int\frac{d^2q}{2\pi}\frac{B(q)}{q^2}[\exp(iq\,.\,r)-1]$$

$\theta(r)\approx$ angle between r and some fixed axis.

$B(q)$ is a very complicated function. However, for any $B(q)$ with both $B(0)=1$ and $B(q)\to 0$ as $q\to\infty$ (to ensure convergence) the behaviour of $L(r)$ is the same. $L(0)$ will be 0 and $L(r)$ will be $\ln(r/A)$ for large r. It is therefore our belief that the exact choice of $B(q)$ does not affect the universality class of our critical line. So the choice $B(q)=J_0(aq)$, a Bessel function, should not change the relations between the critical index expansions. With this choice of $B(q)$, $L(r)$ becomes (exactly)

$$L(r)=\begin{cases}\ln r/a & r\geqslant a\\ 0 & r\leqslant a.\end{cases}$$

Similar considerations compel us to choose $\theta(r)$ to be the mentioned angle exactly.

Once these simple forms for $L(r)$ and $\theta(r)$ have been chosen, the correlation functions can all be calculated. Here are two important correlation functions not easily found elsewhere:

$$\begin{aligned}&\langle \mathrm{F}_{11}(r_1)\mathrm{O}_{n,m}(r_2)\mathrm{O}_{-n,-m}(r_3)\rangle\\ &\qquad=2(n^2-m^2)\langle \mathrm{O}_{n,m}(r_2)\mathrm{O}_{-n,-m}(r_3)\rangle\\ &\qquad\quad\times(\nabla_-L(r_2-r_1)-\nabla_-L(r_3-r_1))(\nabla_+L(r_2-r_1)-\nabla_+L(r_3-r_1))\\ &\langle \mathrm{F}_{11}(r_1)\mathrm{F}_{11}(r_2)\mathrm{O}_{n,m}(r_3)\mathrm{O}_{-n,-m}(r_4)\rangle_c\\ &\qquad=\langle \mathrm{O}_{n,m}(r_3)\mathrm{O}_{-n,-m}(r_4)\rangle[-(n-m)^2\nabla_-^2L(r_2-r_1)(\nabla_+L(r_3-r_1)\\ &\qquad\quad-\nabla_+L(r_4-r_1))(\nabla_+L(r_3-r_2)-\nabla_+L(r_4-r_2))\\ &\qquad\quad-(n+m)^2\nabla_+^2L(r_2-r_1)(\nabla_-L(r_3-r_1)-\nabla_-L(r_4-r_1))(\nabla_-L(r_3-r_2)\\ &\qquad\quad-\nabla_-L(r_4-r_2))+4(n^2-m^2)^2(\nabla_+L(r_3-r_1)\\ &\qquad\quad-\nabla_+L(r_4-r_1))(\nabla_+L(r_3-r_2)-\nabla_+L(r_4-r_2))\\ &\qquad\quad\times(\nabla_-L(r_3-r_1)-\nabla_-L(r_4-r_1))(\nabla_-L(r_3-r_2)-\nabla_-L(r_4-r_2))].\end{aligned}$$

Here $\nabla_\pm=\partial/\partial x\pm \mathrm{i}\partial/\partial y$.

Now we can go on to evaluate the various integrals in each region of space where some fixed combination of the $|r_i-r_j|$ is greater than the cutoff a (as opposed to less than a). The only difficult region is that where all $|r_i-r_j|\geqslant a$. This problem can be overcome by noting that

$$\langle\tilde{\mathrm{O}}(r_1,r_2)\mathrm{E}(r_3)\mathrm{E}(r_4)\rangle_c\equiv\langle\tilde{\mathrm{O}}(r_1,r_2)\mathrm{F}_{11}(r_3)\mathrm{F}_{11}(r_4)\rangle_c$$

in this region for many operators $\tilde{\mathrm{O}}$ of interest. This is a great help because the specific form of the F_{11} correlation functions (derivatives of $L(r)$) enables us to evaluate their integrals exactly.

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References

Baxter R J and Kelland S B 1974 *J. Phys. C: Solid St. Phys.* **7** L403
Enting I G 1975 *J. Phys. A: Math. Gen.* **8** L35
Kadanoff L P 1977 *Phys. Rev. Lett.* **39** 903
—— 1979 *Ann. Phys., NY* **120** 39
Kadanoff L P and Brown A C 1979 *Ann. Phys., NY* **121** 318
Luther A and Peshel I 1975 *Phys. Rev.* B **12** 3908
Pruisken A and Kadanoff L P 1980 *Phys. Rev.* B (in press)
Wegner F 1972 *Phys. Rev.* B **5** 4529
—— 1976 in *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic Press) p 8